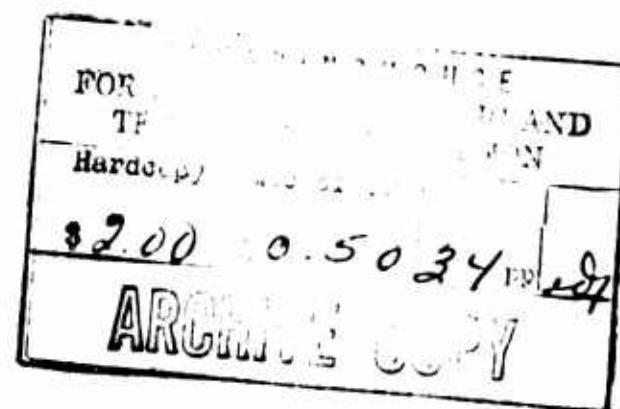


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STOCHASTIC SENSITIVITY ANALYSIS OF MAXIMUM FLOW NETWORKS

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## STOCHASTIC SENSITIVITY ANALYSIS OF MAXIMUM FLOW NETWORKS

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### I. INTRODUCTION

A maximum flow network is defined by a set of arcs and a set of points called nodes. Each arc joins two nodes and has associated with it a positive capacity which represents the maximum amount of flow that may pass over it. One of the nodes is designated as the source and another as the sink. From these nodes, arcs, and capacities the maximum amount of flow that may pass from source to sink may be calculated.

This investigation is concerned with a sensitivity analysis on a class of such networks known as planar networks. Specifically, each arc of the network is subject to anywhere from one to  $n$  breakdowns which result in a reduction in its capacity. The amount of this

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\*\* Ref. 7, p. 7-22.

reduction in capacity is a random variable with known mean and variance.

It is desired to find the smallest possible value of F resulting from,  
at most, n breakdowns where F satisfies:

$$P\{\max \text{ flow} \geq F\} \leq \beta$$

where P stands for probability and  $\beta$  is a small number between zero and one.

## II. MAX-FLOW MIN-CUT THEOREM

The central idea in the solving of maximum flow network problems is summarized in the max-flow min-cut theorem. First, it is necessary to define a cut set and its value.

Definition: Consider a network consisting of nodes that include a source and a sink, and capacitated arcs that join two nodes. Let A and B be a partition of the nodes such that the source is in A and the sink is in B. Then the set of arcs that join a node in A to a node in B is called a cut set and is denoted  $[A, B]$ . Furthermore, the value of this cut set,  $V[A, B]$ , is equal to the sum of the capacities of its arcs.

According to the max-flow min-cut theorem, the maximum flow equals the minimum value of all cut sets.<sup>187</sup>

### III. THE TOPOLOGICAL DUAL

The topological dual of a network, when defined, is another network in which the arcs, instead of having capacities, have lengths. Furthermore, there is a one-to-one correspondence between the proper cuts of the original network and the routes through the dual, and the problem of finding the minimum cut may be reduced to one of finding a shortest route.

Let the original maximum flow network be called the primal. To the primal add an artificial arc extending from the source to the sink and having a capacity of zero. The resulting network will be referred to as the modified primal. The dual network is defined if and only if the primal is source-sink planar, a source-sink planar network being one where the modified primal can be drawn on a sphere in such a way that no two arcs intersect except at a node.

When defined, the dual is constructed in the following manner:

1. Draw the modified primal on a sphere in such a way that no two arcs intersect except at a node.
2. Place a node in each mesh of the modified primal. Let the node in one of the two meshes bounded by the artificial arc be the source and the node in the other of these two meshes be the sink.
3. For each arc except the artificial one construct an arc of the dual that intersects it and joins the nodes in the meshes on either side of it.
4. Assign each arc of the dual a length equal to the capacity of the primal arc it intersects.

Letting a route through the dual be any path from its source to its

sink, it follows that there is a one-to-one correspondence between the proper cuts of the primal and the routes of the dual.<sup>[8]</sup> Specifically, if A is any route through the dual, then the arcs of the primal which intersect A form a proper cut and conversely if B is a proper cut of the primal, then the arcs of the dual which intersect B form a route.

#### IV. PROBLEM FORMULATION

As stated before, the arcs of the maximum flow network in this paper are subject to anywhere from 0 to  $n$  breakdowns. The decrease in capacity of arc  $(a,b)$  due to  $i$  breakdowns is a random variable with unknown distribution but with known mean and variance. The mean and variance of this quantity will be denoted as  $\mu(a,b,i)$  and  $\sigma^2(a,b,i)$  respectively. Furthermore, these distributions are independent for the different arcs. It is desired to find the smallest possible value of  $F$  resulting from, at most,  $n$  breakdowns where  $F$  satisfies:

$$P\{\text{max flow} \geq F\} \leq \beta .$$

This problem can also be formulated in terms of the dual network. Specifically, each arc of the dual is subject to anywhere from 0 to  $n$  improvements. The decrease in length of arc  $(a,b)$  due to  $i$  improvements is a random variable with mean  $\mu(a,b,i)$  and variance  $\sigma^2(a,b,i)$ . It is desired to find the smallest possible value of  $F$  resulting from, at most,  $n$  total improvements on  $n$  or less arcs where  $F$  satisfies:

$$P\{\text{min route} \geq F\} \leq \beta$$

It is the latter formulation that is considered in this paper. Since examples can be constructed to show that different distributions with the same  $\mu(a,b,i)$  and  $\sigma^2(a,b,i)$  can have different  $F$  and different locations for the improvements, the information given is not sufficient to determine a solution. Accordingly, the distribution considered will be the one whose smallest value of  $F$  is maximum.

Suppose that if improvements occur on certain arcs, the length of

a particular route is a random variable  $L$  with mean  $\mu$  and variance  $\sigma^2$ . It follows from Tchebysheff's extended lemma\*

$$P\{L - \mu \geq \epsilon\} \leq \frac{\sigma^2}{\epsilon^2 + \sigma^2}$$

or

$$P\{L \geq \mu + \epsilon\} \leq \frac{\sigma^2}{\epsilon^2 + \sigma^2}$$

If it is required to minimize  $\mu + \epsilon$  under the condition that:

$$\frac{\sigma^2}{\epsilon^2 + \sigma^2} \leq \beta$$

it follows that

$$\epsilon = \sigma \sqrt{\frac{1}{\beta} - 1}$$

and one obtains

$$P\{L \geq \mu + \sigma \sqrt{\frac{1}{\beta} - 1}\} \leq \beta$$

Thus, the quantity  $\mu + \sigma \sqrt{\frac{1}{\beta} - 1}$  plays the role of the length

of the above mentioned route and will be referred to as its effective length. The algorithm to be presented in this chapter finds those arcs upon which the  $n$  improvements must occur in order that the minimum effective length of all routes through the dual be minimized.

In order that this be a valid criteria, the following assumptions are made:

1.  $\mu(a,b,i) < l(a,b)$  and is strictly increasing in  $i$ .
2.  $\mu(a,b,i) - \sigma(a,b,i) \sqrt{\frac{1}{\beta} - 1} > 0$  for  $i \geq 1$ .

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\* Ref. 9, pp. 111-126.

3.  $u(a, b, i) > u(\bar{a}, \bar{b}, \bar{i})$  implies

$$u(a, b, i) - \sigma(a, b, i) \sqrt{\frac{1}{\theta} - 1} > u(\bar{a}, \bar{b}, \bar{i}) - \sigma(\bar{a}, \bar{b}, \bar{i}) \sqrt{\frac{1}{\theta} - 1}.$$

These assumptions assure that the solution to this problem involves exactly  $n$  breakdowns, assure that the effective length of an arc is always greater than zero and generate arguments in favor of the efficiency of the algorithm.

## V. THE ALGORITHM

Let an  $i$ -arc path to node  $a$  be any path from the source to node  $a$  with  $i$  or fewer improvements occurring on its arcs. Each node is assigned  $n + 1$  sets of labels, the set numbers being designated as  $0, 1, \dots, n$ . Each label consists of four components and is denoted  $(\mu, \sigma^2, t, k)_{a,i}^j$  where  $a$  designates the node,  $i$  the set number, and  $j$  the rank within the set. The individual components are denoted as  $\mu_{a,i}^j$ ,  $(\sigma^2)_{a,i}^j$ ,  $t_{a,i}^j$ , and  $k_{a,i}^j$ . The quantities  $\mu_{a,i}^j$  and  $(\sigma^2)_{a,i}^j$  are related to the mean and variance of  $i$ -arc paths to node  $a$  and at termination  $\bar{\mu}_{S,n}^j$  and  $(\sigma^2)_{S,n}^j$  are the mean and variance of the length of the desired path where  $\bar{j} = \max j$  in this set. The components  $t_{a,i}^j$  and  $k_{a,i}^j$  are tracers which are used to find the path itself. Again,  $S$  and  $\bar{S}$  refer to the source and sink respectively.

The algorithm for finding the mean variance of the length of the desired route is as follows:

1. Set  $i = 0$ .
2. Set  $\mu_{S,i}^1 = (\sigma^2)_{S,i}^1 = 0$ ,  $\mu_{a,i}^1 = (\sigma^2)_{a,i}^1 = \infty$  for  $a \neq S$  and  $s = i$ .
3. Consider each  $a$  and all arcs of the form  $(a,b)$ . For each  $j$  consider the quantity:

$$L = \mu_{b,i-s}^j + l(a,b) - \mu(a,b,s)$$

$$+ \sqrt{(\sigma^2)_{b,i-s}^j + \sigma^2(a,b,s)} \sqrt{\frac{1}{8} - 1}$$

and delete from the set  $i$  at node  $a$  all labels  $j'$  which satisfy:

$$\mu_{a,i}^{j'} \geq \mu_{b,i-s}^j + l(a,b) - \mu(a,b,s)$$

$$\mu_{a,i}^{j'} + \sigma_{a,i}^{j'} \sqrt{\frac{1}{\theta} - 1} \geq L$$

with strict inequality for at least one of these. If there is no  $j'$  satisfying:

$$\mu_{a,i}^{j'} \leq \mu_{b,i-s}^j + l(a,b) - \mu(a,b,s)$$

$$\mu_{a,i}^{j'} + \sigma_{a,i}^{j'} \sqrt{\frac{1}{\theta} - 1} \leq L$$

introduce the label  $(\mu, \sigma^2, t, k)_{a,i}^-$  where

$$\mu_{a,i}^- = \mu_{b,i-s}^j + l(a,b) - \mu(a,b,s)$$

$$(\sigma^2)_{a,i}^- = (\sigma^2)_{b,i-s}^j + \sigma^2(a,b,s)$$

$$t_{a,i}^- = b$$

$$k_{a,i}^- = s$$

If any changes result in the labels after examining an arc, re-rank the labels in order of increasing first component. This results in a change in the subscript  $j$  for some labels. In this set,  $j$  now takes on the values  $1, 2, \dots, r$  where  $r$  is the number of labels in set  $i$  at node  $a$ .

4. If  $s = 0$ , go to 5. Otherwise, decrease  $s$  by 1 and go back to 3.

5. Repeat 3 until no changes in the labels of set  $i$  at node  $a$

result.

6. If  $i < n$ , increase  $i$  by 1 and go back to 2. If  $i = n$ , terminate, as  $\bar{j}_{S,n}$  and  $(\sigma^2)_{S,n}^{j_1}$  are the mean and variance of the length of the desired path where  $j$  equals the number of members of set  $n$  at  $\bar{S}$ .

The path itself may be found by the following procedure:

1. Set  $m = 1$ ,  $a_1 = \bar{S}$ , and  $i_1 = n$ .

2. Let  $(\mu, \sigma^2, t, k)_{S,n}^{j_1}$  be the label of highest rank among set  $n$

at  $\bar{S}$ .

3. Set  $k_m = k_{a_m, i_m}^{j_m}$

4. Increase  $m$  by 1. Set  $i_m = i_{m-1} - k_{m-1}$  and  $a_m = t_{a_{m-1}, i_{m-1}}^{j_{m-1}}$ .

If  $a_m = S$ , go to 6. Otherwise, find a label in set  $i_m$  at node  $a_m$  which satisfies \*.

$$\mu_{a_m, i_m}^{j_m} = \mu_{a_{m-1}, i_{m-1}}^{j_{m-1}} - l(a_{m-1}, a_m) + \mu(a_{m-1}, a_m, k_{m-1})$$

$$(\sigma^2)_{a_m, i_m}^{j_m} = (\sigma^2)_{a_{m-1}, i_{m-1}}^{j_{m-1}} - \sigma^2(a_{m-1}, a_m, k_{m-1})$$

5. Go back to 3.

6. Terminate.  $S = a_m, \dots, a_1 = \bar{S}$  with arc  $(a_j, a_{j+1})$  having  $k_j$

improvements is the desired path. The arcs of the primal upon which breakdowns must occur, are those which intersect this path. Specifically,  $k_j$  breakdowns must occur on the primal arc which intersects arc  $(a_j, a_{j+1})$  in this path.

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\* Note that  $\mu(a, b, 0) = \sigma^2(a, b, 0) = 0$ .

## VI. JUSTIFICATION OF THE ALGORITHM

The justification will consist in showing that the algorithm is finite and that at termination the label of highest rank in set  $n$  at  $\bar{S}$  has as its first two components the mean and variance of the length of the  $n$ -arc path to the sink of minimum effective length. Furthermore, the steps of the tracing procedure can be carried out and finds this path.

Lemma 1: Let  $S = a_0, a_1, \dots, a_m$  with  $k_j$  breakdowns occurring on arc  $(a_{j-1}, a_j)$  be an  $i$ -arc path to  $a_m$  such that all of its arcs are distinct. Let the mean and variance of the length of arc  $(a_{j-1}, a_j)$  be  $\mu_j$  and  $\sigma_j^2$  respectively. Then the total length of this path is a random variable whose mean is equal to  $\sum_{j=1}^m \mu_j$  and whose variance is equal to  $\sum_{j=1}^m \sigma_j^2$ .

Proof: The amounts by which the lengths of the different arcs can be reduced are independent. Hence, the lengths of the arcs themselves are independent and their variances may be added to give the variance of the total length of the path. Of course, the means may be added regardless of whether or not the distributions are independent.

Corollary 2: Let  $S = a_0, \dots, a_m$  with  $k_j$  breakdowns occurring on arc  $(a_{j-1}, a_j)$  be a non-cyclic  $i$ -arc path to node  $a_m$ , and let the length of arc  $(a_{j-1}, a_j)$  have mean  $\mu_j$  and variance  $\sigma_j^2$ . Then the length of this path has mean  $\sum_{j=1}^m \mu_j$  and variance  $\sum_{j=1}^m \sigma_j^2$ .

Proof: Since the path is non-cyclic, all its arcs are distinct and the corollary follows from lemma 1.

Definition: The pair  $(\mu, \sigma^2)$  is said to dominate the pair  $(\bar{\mu}, \bar{\sigma}^2)$  if and only if:

$$i) \mu \leq \bar{\mu}$$

$$ii) \mu + \sigma \sqrt{\frac{1}{\beta} - 1} \leq \bar{\mu} + \bar{\sigma} \sqrt{\frac{1}{\beta} - 1}$$

with strict inequality for at least one of these. If path 1 is an i-arc path to node a whose length has mean  $\mu_1$  and variance  $\sigma_1^2$ , and path 2 is an i-arc path to node a whose length has mean  $\mu_2$  and variance  $\sigma_2^2$ , then path 1 is said to dominate path 2 if and only if the pair  $(\mu_1, \sigma_1^2)$  dominates  $(\mu_2, \sigma_2^2)$ . An i-arc path to node a that is not dominated by any other i-arc path to node a is said to be an undominated i-arc path. Furthermore, a label  $(\mu, \sigma^2, t, k)_{a,i}^j$  dominates the label  $(\mu, \sigma^2, t, k)_{a,i}^{\bar{j}}$  if and only if the pair  $(\mu_{a,i}^j, (\sigma^2)_{a,i}^j)$  dominates  $(\mu_{a,i}^{\bar{j}}, (\sigma^2)_{a,i}^{\bar{j}})$ .

Note that the dominance property is transitive, that a label can be introduced into a set only if no other label in that set dominates it; and, finally, when a label is being considered for introduction into a set, the labels that are dropped from that set are precisely those which are dominated by the one being considered.

Lemma 3: A label can not dominate another label in its set.

Proof: Suppose the theorem holds at one stage of the algorithm and that node a and arc (a,b) with s improvements are being considered with respect to the label  $(\mu, \sigma^2, t, k)_{b,i-s}^j$ .

Let :

$$\bar{\mu} = \mu_{b,i-s}^j + l(a,b) - \mu(a,b,s)$$

$$\bar{\sigma}^2 = (\sigma^2)_{b,i-s}^j + \sigma^2(a,b,s)$$

All labels among set  $i$  at node  $a$  whose first two components are dominated by  $(\bar{\mu}, \bar{\sigma}^2)$  are dropped from this set and the condition

still holds. Then the label  $(\mu, \sigma^2, t, k)_{a,i}^j = (\bar{\mu}, \bar{\sigma}^2, b, s)$  is introduced into this set if and only if it is not dominated by any other label in this set and the property is still preserved. The only other way a change in labels can result is through the introduction of  $(0,0,-,-)$  or  $(\infty, \infty, -, -)$  into an empty set. Initially, the conditions of the lemma hold with all sets empty and the lemma follows from induction.

Lemma 4: A label can only be dropped from a set if another label which dominates it is introduced into that set.

Proof: Suppose a label  $(\mu, \sigma^2, t, k)_{a,i}^j$  is to be dropped from set  $i$  at node  $a$ . Then there is a quadruple  $(\bar{\mu}, \bar{\sigma}^2, \bar{t}, \bar{k})$  which will be introduced as a label into this set provided no other label would dominate it and which has the property that it dominates  $(\mu, \sigma^2, t, k)_{a,i}^j$ . Furthermore, no other label in this set can dominate it for if it did, it would also dominate  $(\mu, \sigma^2, t, k)_{a,i}^j$ . Hence,  $(\mu, \sigma^2, t, k)_{a,i}^j = (\bar{\mu}, \bar{\sigma}^2, \bar{t}, \bar{k})$  is introduced into set  $i$  at node  $a$  proving the theorem.

Corollary 5: If a label is dropped from a particular set, there will always be a label in that set which dominates it.

Proof: This follows immediately from Lemma 4 and the transitivity of the dominance property.

Lemma 6: Let path 1,  $s = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  and path 2,  $s = \bar{a}_0, \dots, \bar{a}_m = a$  with  $\bar{k}_j$  improvements on arc  $(\bar{a}_{j-1}, \bar{a}_j)$  be non-cyclic i-arc paths to node  $a$ . Let path 1' and 2' be  $(i+r)$ -arc paths to node  $b$  formed by adding arc  $(a, b)$  with  $r$  improvements to path 1 and path 2 respectively. If path 1 dominates path 2, then path 1' dominates path 2'.

Proof: Since paths 1 and 2 are non-cyclic, it follows that the arcs of each path 1, 2, 1', and 2' are distinct. Letting the pairs  $(\mu(a), \sigma^2(a))$ ,  $(\bar{\mu}(a), \bar{\sigma}^2(a))$ ,  $(\mu(b), \sigma^2(b))$  and  $(\bar{\mu}(b), \bar{\sigma}^2(b))$  be the means and variances of the lengths of paths 1, 2, 1' and 2', it follows from Lemma 1 that:

$$\mu(b) = \mu(a) + l(a, b) - \mu(a, b, r)$$

$$\bar{\mu}(b) = \bar{\mu}(a) + l(a, b) - \mu(a, b, r)$$

$$\sigma^2(b) = \sigma^2(a) + \sigma^2(a, b, r)$$

$$\bar{\sigma}^2(b) = \bar{\sigma}^2(a) + \sigma^2(a, b, r)$$

Therefore:

$$\mu(a) < \bar{\mu}(a)$$

$$\mu(a) + l(a, b) - \mu(a, b, r) < \bar{\mu}(a) + l(a, b) - \mu(a, b, r)$$

$$\mu(b) < \bar{\mu}(b)$$

with equality only if  $\mu(a) = \bar{\mu}(a)$  and the first condition is satisfied.

For the second condition one has:

Case 1:

$$\mu(a) + \sigma(a)\sqrt{\frac{1}{\theta} - 1} = \bar{\mu}(a) + \bar{\sigma}(a)\sqrt{\frac{1}{\theta} - 1}$$

$$\mu(a) < \bar{\mu}(a)$$

$$\sigma(a) > \bar{\sigma}(a)$$

$$\sigma^2(a) - \bar{\sigma}^2(a) = [\sigma^2(a) + \sigma^2(a, b, r)] - [\bar{\sigma}^2(a) + \sigma^2(a, b, r)]$$

$$\sigma^2(a) - \bar{\sigma}^2(a) = \sigma^2(b) - \bar{\sigma}^2(b)$$

$$\sigma(a) + \bar{\sigma}(a) \leq \sigma(b) + \bar{\sigma}(b)$$

$$\sigma(a) - \bar{\sigma}(a) \geq \sigma(b) - \bar{\sigma}(b)$$

$$\mu(a) + [\sigma(a) - \bar{\sigma}(a)]\sqrt{\frac{1}{\theta} - 1} = \bar{\mu}(a)$$

$$\mu(a) + [\sigma(b) - \bar{\sigma}(b)]\sqrt{\frac{1}{\theta} - 1} \leq \bar{\mu}(a)$$

$$\mu(b) + \sigma(b)\sqrt{\frac{1}{\theta} - 1} \leq \bar{\mu}(b) + \bar{\sigma}(b)\sqrt{\frac{1}{\theta} - 1}$$

Case 2:

$$\mu(a) + \sigma(a)\sqrt{\frac{1}{\theta} - 1} < \bar{\mu}(a) + \bar{\sigma}(a)\sqrt{\frac{1}{\theta} - 1}$$

Define  $\sigma$  such that

$$\mu(a) + \sigma\sqrt{\frac{1}{\theta} - 1} = \bar{\mu}(a) + \bar{\sigma}(a)\sqrt{\frac{1}{\theta} - 1}$$

$$\mu(b) + \sqrt{\sigma^2 + \sigma^2(a, b, r)}\sqrt{\frac{1}{\theta} - 1} \leq \bar{\mu}(b) + \bar{\sigma}(b)\sqrt{\frac{1}{\theta} - 1}$$

Also  $\sigma(a) < \sigma$ . Thus:

$$\sigma(b) < \sqrt{\sigma^2 + \sigma^2(a, b, r)}$$

$$\mu(b) + \sigma(b) \sqrt{\frac{1}{8} - 1} < \bar{\mu}(b) + \bar{\sigma}(b) \sqrt{\frac{1}{8} - 1} .$$

Lemma 7: For each label  $(\mu, \sigma^2, t, k)_{a,i}^j$ , the quantities  $\mu_{a,i}^j$  and  $(\sigma^2)_{a,i}^j$  are either infinite or are equal to the mean and variance of an  $i$ -arc path to node  $a$ .

Proof: Suppose at one stage of the algorithm that for each label  $(\mu, \sigma^2, t, k)_{a,i}^j$  where  $\mu_{a,i}^j$  and  $(\sigma^2)_{a,i}^j$  are not infinite, there is an  $i$ -arc path  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  whose arcs are distinct and whose length has mean  $\mu_{a,i}^j$  and variance  $(\sigma^2)_{a,i}^j$  and such that set  $[i - \sum_{j=n+1}^m k_j]$  at node  $a_n$  contains a label whose first two components equals or dominates the mean and variance of the length of the path  $S = a_0, \dots, a_n$  with  $k_j$  breakdowns on arc  $(a_{j-1}, a_j)$ . Suppose the label  $(\mu, \sigma^2, t, k)_{b,i+r}^j$  is introduced through the examination of arc  $(a,b)$  with  $r$  improvements. Then there is a label  $(\mu, \sigma^2, t, k)_{a,i}^j$  such that:

$$\bar{\mu}_{b,i+r}^j = \mu_{a,i}^j + l(a,b) - \mu(a,b,r)$$

$$(\sigma^2)_{b,i+r}^j = (\sigma^2)_{a,i}^j + \sigma^2(a,b,r) .$$

Let  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  be a path which satisfies the above conditions for  $(\mu, \sigma^2, t, k)_{a,i}^j$ . Consider the path  $S = a_0, \dots, a_{m+1} = b$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$ ,  $k_{m+1} = r$ . If each of its arcs is distinct, it satisfies the above

conditions for  $(\mu, \sigma^2, t, k)_{b, i+r}^j$ . Suppose its arcs are not all distinct.

Then there is an  $n < m$  such that  $a_n = a_m$  and  $a_{n+1} = a_{m+1}$ . The

mean and variance of the length of the path  $S = a_0, \dots, a_{n+1}$  with  $k_j$

improvements on arc  $(a_{j-1}, a_j)$  for  $j \leq n$  and  $\sum_{j=n+1}^{m+1} k_j$  improvements on

arc  $(a_n, a_{n+1})$  dominates the pair  $(\mu_{b, i+r}^j, (\sigma^2)_{b, i+r}^j)$ . Also the

path  $S = a_0, \dots, a_n = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$

dominates the path  $S = a_0, \dots, a_m$  with  $k_j$  improvements on arc

$(a_{j-1}, a_j)$ . Therefore, if  $\sum_{j=n+1}^m k_j = 0$ , there is a label in set  $i$  at

node  $a$  which dominates  $(\mu, \sigma^2, t, k)_{a, i}^j$  contradicting lemma 3. On the

other hand, suppose  $\sum_{j=n+1}^m k_j \neq 0$ . Consider any label in set

$[i - \sum_{j=n+1}^m k_j]$  at node  $a$  whose first two components dominate the mean

and variance of the length of  $S = a_0, \dots, a_n = a$  with  $k_j$  improvements

on arc  $(a_{j-1}, a_j)$ . This label and arc  $(a, b)$  with  $\sum_{j=n+1}^{m+1} k_j > r$

improvements will be examined prior to the introduction of  $(\mu, \sigma^2, t, k)_{b, i+r}^j$

as a label. After this examination there must be a label among set  $(i+r)$

at node  $a_{m+1} = b$  which equals or dominates  $(\mu, \sigma^2, t, k)_{b, i+r}^j$ . But this

prevents its introduction as a label. Thus, the arcs of  $S = a_0, \dots, a_{m+1} = b$

are distinct. The introduction of  $(0, 0, -, -)$  into any set at the source

or  $(\infty, \infty, -, -)$  into any set preserves the above properties. Furthermore, the

dropping of a label preserves these properties since it is immediately followed by the introduction of a label into its set which dominates it. Initially these properties hold with all sets of labels being empty. The lemma follows from induction.

Lemma 8: The labeling algorithm is finite.

Proof: Since the first two components of a label are either infinite or are equal to the mean and variance of a path whose arcs are distinct, it follows that this pair must be selected from a finite set. In addition, the choices for the  $t_{a,i}^j$  and  $k_{a,i}^j$  are finite. Thus, the labels themselves are selected from a finite set. Furthermore, no label may be introduced into the same set more than once, for if it is once dropped there is another label in the set which dominates it and prevents its re-entry. Thus, the algorithm is finite.

Lemma 9: At termination no labels are infinite at any node to which a path exists.

Proof: Consider set  $i$  at node  $a$ . Let  $S = a_0, \dots, a_m = a$  be any path to  $a$ . It follows that on or before the  $(m+1)^{st}$  examination of the nodes for set  $i$  in step 5 a finite label will be introduced into the set  $i$  at node  $a$ .

Corollary 10: At termination  $u_{a,i}^j$  and  $(\sigma^2)_{a,i}^j$  correspond to the mean and variance of  $i$ -arc paths to node  $a$  for all  $i$  and all  $j$  provided a path  $a$  exists.

Proof: This follows immediately from lemma 9.

Lemma 11: Let  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc

$(a_{j-1}, a_j)$  be an undominated i-arc path to node  $a$ . Then this path contains no cycles.

Proof: Suppose this path does contain a cycle. Let  $(\bar{a}_{j-1}, \bar{a}_j)$ ,  $j = 1, \dots, \bar{m}$  be the distinct arcs of this path. Furthermore, let arc  $(\bar{a}_{j-1}, \bar{a}_j)$  be used  $c_j$  times in this path and let its length have mean  $\mu_j$  and variance  $\sigma_j^2$ . Then the total length of this path has mean  $\sum_{j=1}^{\bar{m}} c_j \mu_j$  and variance  $\sum_{j=1}^{\bar{m}} c_j^2 \sigma_j^2$ ,  $c_j \geq 1$  all  $j$ . Delete all cycles from this path. The resulting path is an i-arc path to node  $a$  whose length has mean  $\sum_{j=1}^{\bar{m}} \bar{c}_j \mu_j$  and variance  $\sum_{j=1}^{\bar{m}} \bar{c}_j^2 \sigma_j^2$ ,  $\bar{c}_j \leq 1$  all  $j$  and  $\bar{c}_j = 0$  for some  $j$ . Thus, the path  $S = a_0, \dots, a_m = a$  is not undominated.

Lemma 12: Let  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  be an undominated i-arc path to node  $a$ . Then  $S = a_0, \dots, a_{m-1}$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  is an undominated  $(i-k_m)$ -arc path to node  $a_{m-1}$ .

Proof: Suppose the above path is not undominated. Let  $S = b_0, \dots, b_{\bar{m}} = a_{m-1}$  with  $\bar{k}_j$  improvements on arc  $(b_{j-1}, b_j)$  be an undominated  $(i-k_m)$ -arc path to node  $a_{m-1}$  that dominates  $S = a_0, \dots, a_{m-1}$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$ . Then  $S = b_0, \dots, b_{\bar{m}}$  and  $S = a_0, \dots, a_{m-1}$  contain no cycles and it follows from lemma 6 that  $S = b_0, \dots, b_{\bar{m}}, a_m$  with  $\bar{k}_j$  improvements on arc  $(b_{j-1}, b_j)$  and  $k_m$

improvements on arc  $(b_{\bar{m}}, a_m)$  is an  $i$ -arc path to node  $a_m$  which dominates  $S = a_0, \dots, a_m$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  contradicting the hypothesis.

Lemma 13: At termination the set of pairs  $(\mu_{a,i}^j, (\sigma^2)_{a,i}^j)$  is the set of means and variances of all undominated  $i$ -arc paths to node  $a$  provided a path to  $a$  exists.

Proof: Suppose that after the algorithm terminates for the sets  $r$  the lemma holds for all sets of pairs  $(\mu_{a,i}^j, (\sigma^2)_{a,i}^j)$ ,  $i \leq r$ . Let  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  be an undominated  $(r+1)$ -arc path to node  $a$  and let  $(a_{p-1}, a_p)$  be the last arc in this path upon which improvements occur. From repeated application of lemma 12 it follows that  $S = a_0, \dots, a_{p-1}$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  is an undominated  $(r+1-k_p)$ -arc path to node  $a_{p-1}$ .

Let the mean and variance of the lengths of these two paths be the pairs  $(\mu_m^2, \sigma_m^2)$  and  $(\mu_{p-1}^2, \sigma_{p-1}^2)$  respectively. Then there exists a pair

$(\mu_{a_{p-1}, r+1-k_p}^j, (\sigma^2)_{a_{p-1}, r+1-k_p}^j) = (\mu_{p-1}^2, \sigma_{p-1}^2)$  . (i.e., there is a label in set  $r+1-k_p$  at node  $a_{p-1}$  whose first two components are  $\mu_{p-1}$  and  $\sigma_{p-1}^2$ .) Note that the path  $S = a_0, \dots, a_p$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  is an undominated  $(r+1)$ -arc path to node  $a_p$ . Denote the mean and variance of the length of this path by  $\mu_p$  and  $\sigma_p^2$ . There can be no pair  $(\mu_{a_p, r+1}^j, (\sigma^2)_{a_p, r+1}^j)$  that dominates  $(\mu_p^2, \sigma_p^2)$  since such

a pair must be infinite or be equal to the mean and variance of an  $(r+l)$ -arc path to node  $a_p$ . Thus, after examining set  $(r+l-k_p)$  at node  $a_{p-1}$  and arc  $(a_{p-1}, a_p)$  with  $k_p$  improvements, there must be a label  $(\mu, \sigma^2, t, k)_{a_p, r+l}^j$  such that  $\mu_{a_p, r+l}^j = \mu_p$  and  $(\sigma^2)_{a_p, r+l}^j = \sigma_p^2$ . Furthermore, this label can never be dropped. Repeated application of this argument yields the conclusion that eventually a label  $(\mu, \sigma^2, t, k)_{a, r+l}^j$  will be introduced where  $\mu_{a, r+l}^j = \mu_m$  and  $(\sigma^2)_{a, r+l}^j = \sigma_m^2$  and that it will never be dropped. Furthermore, any label in set  $r+l$  at node  $a$  whose first two components correspond to the mean and variance of a dominated  $(r+l)$ -arc path to node  $a$  will be dropped by the introduction of a label whose first two components correspond to the mean and variance of the length of a path that dominates the above mentioned one. Thus, when the algorithm terminates for the sets  $r+l$ , the lemma holds for all sets of pairs  $(\mu_{a,i}^j, (\sigma^2)_{a,i}^j)$ ,  $i \leq r+l$ . Let  $S = b_0, \dots, b_m = a$  be an undominated 0-arc path to node  $a$ . An argument similar to that above shows that after, at most,  $m + 1$  checks of the nodes there is a label  $(\mu, \sigma^2, t, k)_{a, 0}^j$  such that  $\mu_{a, 0}^j$  and  $(\sigma^2)_{a, 0}^j$  are the mean and variance of the length of this path; and, finally, when the algorithm terminates for the sets 0, the lemma holds for all sets of pairs  $(\mu_{a, 0}^j, (\sigma^2)_{a, 0}^j)$ . The lemma itself follows from induction.

Theorem 14: When the algorithm terminates, the label of highest rank among set  $n$  at  $\bar{S}$  has as its first two components the mean and

variance of the n-arc path of minimum effective length to  $\bar{S}$ .

Proof: The n-arc path of minimum effective length is undominated and therefore there must exist a label  $(\mu, \sigma^2, t, k)_{\bar{S}, n}^j$  at termination such that  $\mu_{\bar{S}, n}^j$  and  $(\sigma^2)_{\bar{S}, n}^j$  correspond to the mean and variance of the length of this path. Furthermore, there can be no other label  $(\mu, \sigma^2, t, k)_{\bar{S}, n}^{\bar{j}}$  of higher rank, for if so one has:

$$\mu_{\bar{S}, n}^j < \mu_{\bar{S}, n}^{\bar{j}}$$

$$\mu_{\bar{S}, n}^j + \sigma_{\bar{S}, n}^j \sqrt{\frac{1}{\beta} - 1} < \mu_{\bar{S}, n}^{\bar{j}} + \sigma_{\bar{S}, n}^{\bar{j}} \sqrt{\frac{1}{\beta} - 1}$$

contradicting lemma 3.

Of course the n-arc path of minimum effective length to  $\bar{S}$  is the route of smallest effective length possible from source to sink due to, at most, n breakdowns. Thus, the algorithm does find the mean and variance of the desired path. It now must be shown that the tracing procedure actually finds this path and the arcs upon which the n improvements must occur.

Lemma 15: Let  $(\mu, \sigma^2, t, k)_{a, i}^j$ ,  $a \neq S$  be a label at termination.

Then there is another label  $(\mu, \sigma^2, t, k)_{b, \bar{i}}^{\bar{j}}$  at termination such that:

$$b = t_{a, i}^j$$

$$\bar{i} = i - k_{a, i}^j$$

$$\mu_{a, i}^j = \mu_{b, \bar{i}}^{\bar{j}} + l(a, b) - \mu(a, b, k_{a, i}^j)$$

$$(\sigma^2)_{a, i}^j = (\sigma^2)_{b, \bar{i}}^{\bar{j}} + \sigma^2(a, b, k_{a, i}^j)$$

Proof: Consider the label from which  $(\mu, \sigma^2, t, k)_{a_i}^j$  was obtained.

This label must satisfy the above conditions, i.e., must be  $(\mu, \sigma^2, t, k)_{b_i}^{j'}$ .

Furthermore, this label can not be dropped, for if it's another label

$(\mu, \sigma^2, t, k)_{b_i}^{j'}$  is introduced which dominates it. Examining this label

and arc  $(a, b)$  with  $k_{a_i}^j$  improvements produces a label which dominates

and hence drops  $(\mu, \sigma^2, t, k)_{a_i}^j$ .

Lemma 16: The tracing procedure is finite.

Proof: The sequence of labels found is strictly decreasing lexicographically in  $(\mu_{a_i}^j, i)$  and hence none can be repeated. Thus, the process is finite.

Lemma 17: Let the sequence of labels found by the tracing procedure be:

$$(\mu, \sigma^2, t, k)_{a_r, i_r}^{j_r}, r = 1, \dots, m$$

Then:

$$\mu_{S,n}^{j_1} - \mu_{a_1, i_1}^{j_1} = \sum_{r=1}^{m-1} [l(a_r, a_{r+1}) - \mu(a_r, a_{r+1}, k_r)]$$

$$(\sigma^2)_{S,n}^{j_1} - (\sigma^2)_{a_1, i_1}^{j_1} = \sum_{r=1}^{m-1} \sigma^2(a_r, a_{r+1}, k_r)$$

where  $k_r = k_{a_r, i_r}^{j_r}$ .

Proof: It follows from the rules of the tracing procedure that:

$$(i) \mu_{a_r, i_r}^{j_r} - \mu_{a_{r+1}, i_{r+1}}^{j_{r+1}} = l(a_r, a_{r+1}) - \mu(a_r, a_{r+1}, k_r), r = 1, \dots, m-1$$

$$\mu_{S, i_m}^{j_m} = \mu_{a_m, i_m}^{j_m} = 0$$

$$(ii) (\sigma^2)_{a_r, i_r}^{j_r} - (\sigma^2)_{a_{r+1}, i_{r+1}}^{j_{r+1}} = \sigma^2(a_r, a_{r+1}, k_r), r = 1, \dots, m-1$$

$$(\sigma^2)_{S, i_m}^{j_m} = (\sigma^2)_{a_m, i_m}^{j_m} = 0.$$

Summing the equations (i) gives the first relationship and summing the equations (ii) gives the second.

Theorem 18: Let the sequence of labels found by the tracing procedure be as in lemma 17. Then  $S = a_m, \dots, a_1 = \bar{S}$  with  $k_r = k_{a_r, i_r}^{j_r}$  improvements on arc  $(a_r, a_{r+1})$  is an n-arc path from source to sink whose length has mean  $\mu_{S,n}^{j_1}$  and variance  $(\sigma^2)_{S,n}^{j_1}$ .

Proof: The quantities  $\mu_{S,n}^{j_1}$  and  $(\sigma^2)_{S,n}^{j_1}$  equal the sums of the means and variances, respectively, of the arcs in this path. All the arcs in this path are distinct, for if not there would be a cycle; and, deleting all cycles would yield an n-arc path to  $\bar{S}$  such that its length has mean strictly less than  $\mu_{S,n}^{j_1}$  and variance not exceeding  $(\sigma^2)_{S,n}^{j_1}$  contradicting theorem 14. Thus, the theorem follows from lemma 1.

This completes the justification of the procedure for finding the desired route itself. The primal arcs upon which breakdowns must occur

are those which intersect this route. In particular,  $k_j$  breakdowns must occur upon that arc of the primal which intersects arc  $(a_j, a_{j+1})$ .

### VII. NORMAL CASE

Suppose it is known that the decrease in the capacities of the arcs due to breakdowns are independently and normally (or rather approximately normally) distributed with known mean and variance. It follows that for any breakdown pattern all routes through the dual are normally distributed.

Let the length of a particular path be a random variable  $L$  whose distribution is normal with mean  $\mu$  and variance  $\sigma^2$ .

Then:

$$P \left\{ \frac{L - \mu}{\sigma} \geq \Phi(1-\theta) \right\} = \theta$$

where  $\Phi(1-\theta)$  satisfies

$$P \left\{ Y \leq \Phi(1-\theta) \right\} = 1 - \theta$$

where  $Y$  is standard normal. Thus:

$$P \left\{ L \geq \mu + \sigma\Phi(1 - \theta) \right\} = \theta$$

Hence,  $\mu + \sigma\Phi(1 - \theta)$  plays the role of the effective length of this path. Suppose it is desired to find the n-arc path to the sink of minimum effective length. Then the algorithm just presented may be

used to solve this problem by replacing  $\sqrt{\frac{1}{\theta} - 1}$  with  $\Phi(1 - \theta)$ .

### VIII. PRACTICALITY OF THE ALGORITHM

Since this algorithm is in many ways similar to the minimum route algorithm\* which is highly efficient, it would seem that this algorithm would be practical, provided the number of labels in each set remained small. While examples have not been solved to establish whether or not the number of labels in each set is likely to remain small, there are indications in favor of this happening.

The first of these indications is the assumption that:

$$\mu(a,b,i) > \mu(\bar{a},\bar{b},\bar{i})$$

implies

$$\mu(a,b,i) - \sigma(a,b,i) \sqrt{\frac{1}{8} - 1} > \mu(\bar{a},\bar{b},\bar{i}) - \sigma(\bar{a},\bar{b},\bar{i}) \sqrt{\frac{1}{8} - 1} .$$

Thus, the difference between the zero-breakdown length of a path and its effective length tends to increase as the mean decrease in length due to breakdowns increases.

The second of these indications is even more convincing. Let A and B be two paths to node a such that A has smaller mean length than B but does not dominate it. Then if arcs with breakdowns are added to the two paths, the new path formed from A tends to dominate that formed from B. This is summarized in theorem 1.

Theorem 1: Consider the following quantities:

$$\bar{\mu}_1 = \mu_1 + \mu$$

$$\bar{\mu}_2 = \mu_2 + \mu$$

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\*Ref. 7, pp. 130-134.

$$L_1 = \mu_1 + |\sigma_1| \sqrt{\frac{1}{\theta} - 1}$$

$$L_2 = \mu_2 + |\sigma_2| \sqrt{\frac{1}{\theta} - 1}$$

$$\bar{L}_1 = \mu_1 + \mu + \sqrt{\sigma_1^2 + \sigma^2} \sqrt{\frac{1}{\theta} - 1}$$

$$\bar{L}_2 = \mu_2 + \mu + \sqrt{\sigma_2^2 + \sigma^2} \sqrt{\frac{1}{\theta} - 1}$$

Then if  $\sigma_1^2 > \sigma_2^2$ ,  $\bar{\mu}_1 - \bar{\mu}_2 = \mu_1 - \mu_2$  and  $\bar{L}_1 - \bar{L}_2 \leq L_1 - L_2$  with equality if and only if  $\sigma^2 = 0$ .

Proof:

$$\bar{\mu}_1 - \bar{\mu}_2 = \mu_1 - \mu_2 \text{ follows trivially}$$

$$(\bar{L}_1 - \bar{L}_2) - (L_1 - L_2)$$

$$= \left[ \sqrt{\sigma_1^2 + \sigma^2} - \sqrt{\sigma_2^2 + \sigma^2} - |\sigma_1| + |\sigma_2| \right] \sqrt{\frac{1}{\theta} - 1}$$

$$|\sigma_1| > |\sigma_2|$$

$$\sigma_1^2 + \sigma^2 - \sigma_1^2 = \sigma_2^2 + \sigma^2 - \sigma_2^2$$

$$\sqrt{\sigma_1^2 + \sigma^2} + |\sigma_1| > \sqrt{\sigma_2^2 + \sigma^2} + |\sigma_2|$$

$$\sqrt{\sigma_1^2 + \sigma^2} - |\sigma_1| < \sqrt{\sigma_2^2 + \sigma^2} - |\sigma_2| \text{ if } \sigma^2 \neq 0$$

$$\sqrt{\sigma_1^2 + \sigma^2} - \sqrt{\sigma_2^2 + \sigma^2} - |\sigma_1| + |\sigma_2| < 0$$

$$(\bar{L}_1 - \bar{L}_2) - (L_1 - L_2) < 0$$

Note that equality holds in the above if  $\sigma^2 = 0$ .

The significance in this is that it tends to keep the number of labels in a set from increasing at a rapid rate as one gets further from the source.

LIST OF SYMBOLS

(a,b)	Arc joining a and b
l(a,b)	Length of arc joining a and b
$\mu(a,b,i)$	Mean of the decrease in capacity of (a,b) due to i breakdowns
$\sigma^2(a,b,i)$	Variance of the decrease in capacity of (a,b) due to i breakdowns
$(\mu, \sigma^2, t, k)_{a,i}^j$	A label for the stochastic case
$\mu_{a,i}^j$	First component of $(\mu, \sigma^2, t, k)_{a,i}^j$
$(\sigma^2)_{a,i}^j$	Second component of $(\mu, \sigma^2, t, k)_{a,i}^j$
$t_{a,i}^j$	Third component of $(\mu, \sigma^2, t, k)_{a,i}^j$
$k_{a,i}^j$	Fourth component of $(\mu, \sigma^2, t, k)_{a,i}^j$
[A,B]	Cut set [A,B]
v[A,B]	Value of cut set [A,B]

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